

Thick domain walls in a polynomial approximation ^{*}

by

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Abstract

Relativistic domain walls are studied in the framework of a polynomial approximation to the field interpolating between different vacua and forming the domain wall. In this approach we can calculate evolution of a core and of a width of the domain wall. In the single, cubic polynomial approximation used in this paper, the core obeys Nambu-Goto equation for a relativistic membrane. The width of the domain wall obeys a nonlinear equation which is solved perturbatively. There are two types of corrections to the constant zeroth order width: the ones oscillating in time, and the corrections directly related to curvature of the core. We find that curving a static domain wall is associated with an increase of its width. As an example, evolution of a toroidal domain wall is investigated.

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1 Introduction

Recently one observes rapidly growing interest in time evolution of topological defects in 3+1 dimensional space-time. This subject is important for many branches of physics. Without attempting to present here a complete list, let us mention vortices in superconductors [1] and in superfluids [2], defects in liquid crystals [3], magnetic domain walls [4], cosmic strings [5], [6], and a flux tube in QCD [7]. While evolution of topological defects in 1+1 dimensional space-time has been rather well understood, in 3+1 dimensions relatively little is known, and the problem is actually a formidable one.

The type of equations from which one attempts to calculate evolution of topological defects depends in an essential manner on the physical context. In condensed matter physics one uses e.g. diffusion type equations [8] or nonlinear Schrödinger type equations [2]. For cosmic strings in a negligible gravitational field, or for the particle physics flux-tubes, one should use Poincaré invariant wave equations.

Our paper is devoted to dynamics of domain walls governed by a Poincaré invariant wave equation. In this case, several analytical approaches have been made, see e.g. [9], [10], [11], [12] as well as numerical calculations, see, e.g. [13], [14]. They have given insights into the dynamics of domain walls, and have revealed the richness and intricacy of it. Our motivation for studying the dynamics of relativistic domain walls is of rather mathematical character – relative simplicity of the pertinent field equations makes them a convenient testing ground for new methods of calculating the evolution of topological defects. Nevertheless, direct physical applications are also possible. Relativistic domain walls appear in a field-theoretic approach to cosmology, and in particle physics (e.g. the surface of a quark bag can be regarded as a domain wall). Moreover, the polynomial approximation we develop in this paper can also be applied to non-relativistic domain walls observed in condensed matter physics – no significant changes are required.

Till now, the main line of the analytical approaches to the description of dynamics of a domain wall has been to reduce the initial, 3+1 dimensional field-theoretical system to an effective theory of a classical, relativistic membrane. This approach, called the effective action method, is very appealing conceptually, and it is correct in principle. However, because of the complexity of the pertinent field-theoretic equations, it is rather difficult to carry out the necessary calculations without shortcuts, which in turn introduce some

uncertainty about the final result. For a critical discussion of the effective action method we refer the reader to the paper [15]. For the most recent application of the effective action method to domain walls, see [16].

Our opinion is that at the present stage of the subject one should develop and refine various methods of investigations of the dynamics of the domain walls. It seems that because of enormous complexity of the full, 3+1 dimensional, nonlinear field-theoretic dynamics involved, it is long time until we can analytically or numerically calculate evolution of a generic domain wall without any difficulty.

In the present paper we generalize the method of analysis of evolution of relativistic domain walls proposed in [17]. The main characteristic feature of this approach is a simple, approximate polynomial Ansatz for the field inside the domain wall, while outside of it the field has the exact vacuum values. The coefficients of this polynomial are calculated from the field equation and from boundary conditions. Actually, this approximation can be regarded as an application of splines, [18]. In the paper [17] this approach has been applied only to cylindrical and spherical domain walls. In the present paper we apply the polynomial approximation to generic smooth domain walls - this is presented in Section 2. We approximate the field inside the domain wall by a cubic polynomial in a transverse, co-moving coordinate. We obtain Nambu-Goto type equation for a core of the domain wall, and a nonlinear equation for the width of the domain wall. Also, a perturbative scheme for solving the latter equation is presented. It is based on dividing the width into two components: the one oscillating with characteristic frequency given by the mass of the scalar field, and the other one directly related to curvature of the core.

In the next Section we rewrite the Nambu-Goto equation, and a formula for the width of the domain wall, in terms of local curvature radii and velocity perpendicular to the core. This gives a rather nice and useful insight into the dynamics of the domain wall. We show that a force acting on a small piece of the core can be regarded as being due to a surface tension. We also show that curving a static domain wall is associated with an increase of its width.

In Section 4 we calculate the energy of the domain wall. We find that the energy density depends on the curvature of the core, and that nonuniformities of the width increase the energy density. We also find, rather surprisingly, that curving the domain wall seems to decrease its energy. These results, as well as all others in this paper, are obtained for slightly curved domain walls;

our approximations break down when the curvature radii become comparable with the width of the domain wall.

Next, in Section 5, we consider as an example evolution of a toroidal domain wall. We find two types of evolution of this domain wall.

Section 6 contains a suggestion how to improve our approximate Ansatz for the scalar field, and also some ending remarks.

In the Appendix we discuss accuracy of the polynomial approximation.

2 General formalism for evolution of the domain wall in the polynomial approximation

In this subsection we would like to generalize the formalism presented in [17], where only cylindrical and spherical domain walls were considered, and to present the corresponding, approximate solution of the field equation.

We will investigate domain walls in the well-known model, see e.g. [6], involving only a single, real scalar field Φ with the following Lagrangian

$$L = -\frac{1}{2}\eta_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - \frac{\lambda}{2}\left(\Phi^2 - \frac{M^2}{4\lambda}\right)^2, \quad (1)$$

where $(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$, and λ, M are positive constants. The corresponding field equation is

$$\partial_\mu\partial^\mu\Phi - 2\lambda\left(\Phi^2 - \frac{M^2}{4\lambda}\right)\Phi = 0. \quad (2)$$

The vacuum values of the field Φ are equal to $\pm\Phi_0$, where $\Phi_0 \equiv M/2\sqrt{\lambda}$. The domain wall arises if at a given time the field is equal to one of the two vacuum values in some region of the space, and is equal to the other vacuum value in the complementary part of the space, except for the border layer between the two regions (the domain wall), where the field smoothly interpolates between the vacuum values. It is clear that at each instant of time the field Φ vanishes somewhere inside the border layer. We assume that the locus of these zeros is a smooth surface S . We shall call it the core of the domain wall. The well-known example of the domain wall, with the static core given by the (x^1, x^2) plane, is given by the following exact, static

solution of Eq.(2)

$$\Phi = \Phi_0 \tanh\left(\frac{x^3}{2l_0}\right), \quad (3)$$

where $l_0 \equiv M^{-1}$. The width of this domain wall is of the order l_0 , and energy density is exponentially localised around the (x^1, x^2) plane.

For a generic domain wall, space-time parametrisation of the world-volume Σ of the core (a 3-dimensional manifold embedded in Minkowski space-time, whose time slices coincide with S) can be chosen as follows

$$(X^\mu)(\tau, \sigma^1, \sigma^2) = \begin{pmatrix} \tau \\ X^1(\tau, \sigma^1, \sigma^2) \\ X^2(\tau, \sigma^1, \sigma^2) \\ X^3(\tau, \sigma^1, \sigma^2) \end{pmatrix}, \quad (4)$$

where τ coincides with the laboratory frame time x^0 , and σ^1, σ^2 parametrise the core S at each instant of time.

As usual, we introduce a special coordinate system $(\tau, \sigma^1, \sigma^2, \xi)$ in a vicinity of the world-volume Σ , co-moving with the core, [9]. The new coordinates $(\tau, \sigma^1, \sigma^2, \xi)$ are defined by the following formula

$$x^\mu = X^\mu(\tau, \sigma^1, \sigma^2) + \xi n^\mu(\tau, \sigma^1, \sigma^2), \quad (5)$$

where x^μ are Cartesian lab-frame coordinates in Minkowski space-time, and (n^μ) is a normalised space-like four-vector, orthogonal to the Σ (in the co-variant sense) i.e.

$$n_\mu X_{,a}^\mu = 0, \quad n_\mu n^\mu = 1, \quad (6)$$

where $a = 0$ corresponds to τ ; $a = 1, a = 2$ correspond to σ^1, σ^2 , and $X_{,\tau}^\mu \equiv \partial X^\mu / \partial \tau$, etc. The four-vectors $X_{,\tau}, X_{,\sigma^1}, X_{,\sigma^2}$ are tangent to Σ . For points lying on the core $\xi = 0$, and the parameter τ coincides with the lab-frame time x^0 . For $\xi \neq 0$ τ is not equal to the lab-frame time x^0 . The advantage of using the co-moving coordinates is that the world-volume Σ is described by the simple condition $\xi = 0$. Notice that the definition (5) implies that ξ is a Lorentz scalar.

The next step is to write Eq.(2) in the new coordinates. It is convenient to introduce extrinsic curvature coefficients K_{ab} and induced metrics g_{ab} on Σ :

$$K_{ab} = n_\mu X_{,ab}^\mu, \quad g_{ab} = X_{,a}^\mu X_{\mu,b}, \quad (7)$$

where $a, b = 0, 1, 2$. The covariant metric tensor in the new coordinates can be readily calculated, and it can be written in the following form

$$[G_{\alpha\beta}] = \begin{bmatrix} G_{ab} & 0 \\ 0 & 1 \end{bmatrix}, \quad (8)$$

where $\alpha, \beta = 0, 1, 2, 3$; $\alpha = 3$ corresponds to the ξ coordinate; and

$$G_{ab} = M_{ac}g^{cd}M_{db}, \quad M_{ac} \equiv g_{ac} - \xi K_{ac}. \quad (9)$$

Thus, $G_{\xi\xi} = 1$, $G_{\xi a} = 0$ ($a=0,1,2$, as in (6)). It follows from formula (9) that $\sqrt{-G}$, where $G \equiv \det[G_{\alpha\beta}]$, is given by the formula

$$\sqrt{-G} = -(\sqrt{-g})^{-1} \det M, \quad (10)$$

where as usual $g \equiv \det[g_{ab}]$. $\det M$ can be explicitly evaluated in terms of K_{ab} and g_{ab} :

$$\det M = h(\tau, \sigma^1, \sigma^2, \xi) \ g(\tau, \sigma^1, \sigma^2), \quad (11)$$

where

$$h(\tau, \sigma^1, \sigma^2, \xi) = 1 - \xi K_a^a + \frac{1}{2}\xi^2(K_a^a K_b^b - K_a^b K_b^a) - \frac{1}{3!}\xi^3 \epsilon_{abc} \epsilon^{def} K_d^a K_e^b K_f^c. \quad (12)$$

In formula (11) we have noted that g can depend on τ, σ^1, σ^2 . Thus,

$$\sqrt{-G} = \sqrt{-g} \ h. \quad (13)$$

For raising and lowering the latin indices of the extrinsic curvature coefficients we use the induced metric tensors g_{ab} , g^{ab} . The inverse metric tensor $G^{\alpha\beta}$ is given by

$$[G^{\alpha\beta}] = \begin{bmatrix} G^{ab} & 0 \\ 0 & 1 \end{bmatrix}, \quad (14)$$

where

$$G^{ab} = (M^{-1})^{ac} g_{cd} (M^{-1})^{db}. \quad (15)$$

Simple algebraic calculation gives explicit formula for $(M^{-1})^{ac}$:

$$\begin{aligned} (M^{-1})^{ac} = \frac{1}{h} \Big\{ & g^{ac} [1 - \xi K_b^b + \frac{1}{2}\xi^2(K_b^b K_d^d - K_b^d K_d^b)] \\ & + \xi(1 - \xi K_b^b) K^{ac} + \xi^2 K_d^a K^{dc} \Big\} \end{aligned} \quad (16)$$

(this is just the matrix inverse to $[M_{ab}]$; by definition it has upper indices). In general, the coordinates $(\tau, \sigma^1, \sigma^2, \xi)$ are defined locally, in a vicinity of the world-volume Σ . The allowed range of the ξ coordinate can be determined from the condition $h > 0$. Detailed discussion of the region of validity of the co-moving coordinates has been given in [17]. In the co-moving coordinates the field equation (2) has the following form

$$\frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^a} (\sqrt{-G} G^{ab} \frac{\partial \Phi}{\partial u^b}) + \frac{1}{h} \partial_\xi (h \partial_\xi \Phi) - 2\lambda (\Phi^2 - \frac{M^2}{4\lambda}) \Phi = 0. \quad (17)$$

The basic feature of our approach is the approximate cubic polynomial Ansatz for the Φ field inside the domain wall. We assume that

$$\Phi(\tau, \sigma^1, \sigma^2, \xi) = \begin{cases} +\Phi_0 & \text{for } \xi \geq \xi_0, \\ A\xi + \frac{1}{2}B\xi^2 + \frac{1}{3!}C\xi^3 & \text{for } -\xi_1 \leq \xi \leq \xi_0, \\ -\Phi_0 & \text{for } \xi \leq -\xi_1. \end{cases} \quad (18)$$

Here ξ_0, ξ_1, A, B and C are as yet unknown functions of τ, σ^1 and σ^2 . We assume that $\xi_0, -\xi_1$ lie in the allowed range of the ξ coordinate - roughly speaking this is true when the width of the domain wall is small in comparison with radii of curvature of the core in the local rest-frame of the considered piece of the core.

Notice that $\pm\Phi_0$ are exact solutions of Eq.(2). They are defined also outside of the region of validity of the co-moving coordinates. Therefore, formula (18) actually defines the field Φ in the whole space-time.

The Ansatz (18) implies that in fact we introduce boundaries of the domain wall. The outer boundary $\vec{X}_{(+)}$ in the co-moving reference frame is defined by the formula

$$\vec{X}_{(+)}(\tau, \sigma^1, \sigma^2) = \vec{X}(\tau, \sigma^1, \sigma^2) + \xi_0(\tau, \sigma^1, \sigma^2) \vec{n}(\tau, \sigma^1, \sigma^2),$$

while for the inner one ($\vec{X}_{(-)}$)

$$\vec{X}_{(-)}(\tau, \sigma^1, \sigma^2) = \vec{X}(\tau, \sigma^1, \sigma^2) - \xi_1(\tau, \sigma^1, \sigma^2) \vec{n}(\tau, \sigma^1, \sigma^2).$$

Here \vec{n} is the spatial part of the four-vector (n^μ) .

Inserting the cubic polynomial into Eq.(17) and equating to zero terms proportional to the zeroth and first powers of ξ we obtain the following recurrence relations:

$$B = AK_a^a, \quad (19)$$

$$C = -\square^{(3)}A + (K_b^a K_a^b - 2\lambda\Phi_0^2)A + K_a^a B, \quad (20)$$

where

$$\square^{(3)} \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial u^a} (\sqrt{-g} g^{ab} \frac{\partial}{\partial u^b}) \quad (21)$$

is the three-dimensional d'Alembertian on the world-volume Σ of the core. Of course, the cubic polynomial (18) does not obey Eq.(17) exactly. The leftover terms in Eq.(17) are of the order ξ^2 and higher. We assume that these terms are not important. We will discuss the problem of accuracy of our approximation in the Appendix.

We also require that the field Φ is continuous everywhere, in particular at the boundaries, i.e. for $\xi = \xi_0, \xi = -\xi_1$. Then, by a standard reasoning, we deduce from Eq.(17) that also $\partial_\xi \Phi$ is continuous at the boundaries. The second derivative $\partial_\xi^2 \Phi$ is not continuous at the boundary, in general. The conditions of continuity of Φ and $\partial_\xi \Phi$ at $\xi = \xi_0$ and $\xi = -\xi_1$ give

$$\xi_0 = \xi_1, \quad (22)$$

$$A = \frac{3}{2} \frac{\Phi_0}{\xi_0}, \quad B = 0, \quad C = -\frac{3\Phi_0}{\xi_0^3}. \quad (23)$$

The condition $B = 0$ together with relation (19) gives the equation of motion for the core S :

$$K_a^a = 0. \quad (24)$$

It can be shown that this equation is equivalent to Nambu-Goto equation

$$\square^{(3)} X^\mu = 0 \quad (25)$$

for a relativistic membrane. Thus, in the approximations we have made, the core can be regarded as the Nambu-Goto type relativistic membrane.

Relations (20), (23) give also the following 2+1 dimensional, nonlinear wave equation

$$\square^{(3)} \tilde{A} + \left(\frac{1}{2l_0^2} - K_a^b K_b^a \right) \tilde{A} - \frac{1}{2l_0^2} \tilde{A}^3 = 0 \quad (26)$$

for $\tilde{A}(\tau, \sigma^1, \sigma^2) \equiv \frac{2l_0}{\xi_0}$. \tilde{A} is proportional to the inverse width of the domain wall ($2\xi_0$) measured in the natural length unit l_0 , and it can be regarded as a scalar field defined on the core of the domain wall. Let us note here that $2\xi_0$ is the width in the co-moving coordinates. Its transformation to the

lab-frame is not trivial: it includes Lorentz contraction and other changes. This transformation has been discussed in detail in the paper [17] in the case of cylindrical domain wall.

Eq.(26) can be written in terms of dimensionless variables: introducing dimensionless $\tilde{\tau} \equiv \tau/l_0$ and $\tilde{\sigma}^{1,2} \equiv \sigma^{1,2}/l_0$, $\tilde{\partial}_0 \equiv \partial/\partial\tilde{\tau}$, etc., we have

$$\square^{(3)}\tilde{A} + \left(\frac{1}{2} - l_0^2 K_a^b K_b^a\right)\tilde{A} - \frac{1}{2}\tilde{A}^3 = 0. \quad (27)$$

Because g^{ab} as well as $l_0^2 K_b^a K_a^b$ are dimensionless, all coefficients and variables in Eq.(27) are dimensionless.

Certain solutions of equation (27) for \tilde{A} can be found in an approximation scheme which is a generalization of a perturbative expansion proposed in [17] in the case of cylindrical and spherical domain walls. In that paper one can find a detailed motivation for the subsequent steps; in the present paper we will only briefly describe the scheme. Our scheme probably gives only a certain class of solutions - there might be other solutions which can not be obtained in this way. Roughly speaking, the idea is to restrict our considerations to the cases such that the dimensionless extrinsic curvature $l_0^2 K_b^a K_a^b$ is close to zero, and to expand \tilde{A} in non-negative powers of it. In the zeroth order approximation Eq.(27) has the constant solution

$$\tilde{A}^{(0)} = 1. \quad (28)$$

There are also other solutions in this order, e.g., the ones having form of waves propagating along the core. Solutions of this latter type are time-dependent and have a characteristic frequency of oscillations ≥ 1 , where 1 is the (dimensionless) mass of the \tilde{A} field. Notice that this mass coincides with the mass of the Φ field in the unit l_0^{-1} . We assume also that $l_0^2 K_a^b K_b^a$ and its derivatives are smooth functions of $\tilde{\tau}, \tilde{\sigma}^1, \tilde{\sigma}^2$, and such that the derivatives $\tilde{\partial}_b(l_0^2 K_a^b K_b^a)$ are much smaller than the function $l_0^2 K_a^b K_b^a$ itself. In this case the term $l_0^2 K_a^b K_b^a \tilde{A}$ present in Eq.(27) generates a smooth, non-oscillating component N in \tilde{A} . We introduce the following perturbative Ansatz for \tilde{A} :

$$\tilde{A} = 1 + \Omega + N, \quad (29)$$

where Ω denotes the oscillating component – the amplitude of these oscillations is assumed to be of the order $l_0^2 K_a^b K_b^a$. Equation (27) can be split into

two equations:

$$\square^{(3)}\Omega = (1 + l_0^2 K_a^b K_b^a + 3N + \frac{3}{2}N^2)\Omega + \frac{3}{2}N\Omega^2 + \frac{3}{2}\Omega^2 + \frac{1}{2}\Omega^3, \quad (30)$$

$$N = \square^{(3)}N - l_0^2 K_a^b K_b^a - l_0^2 K_a^b K_b^a N - \frac{3}{2}N^2 - \frac{1}{2}N^3. \quad (31)$$

The term $\square^{(3)}N$ is by assumption regarded as small in comparison with the zeroth order contribution to N . Therefore, eq.(31) implies the following result for the lowest order contribution $N^{(1)}$ to N :

$$N^{(1)} = -l_0^2 K_a^b K_b^a. \quad (32)$$

From Eq.(30) we obtain the equation for the zeroth order oscillating contribution $\Omega^{(1)}$ to Ω :

$$\square^{(3)}\Omega^{(1)} = \Omega^{(1)}. \quad (33)$$

Detailed form of this equation depends on the metric g_{ab} . Because the frequency of oscillations implied by Eq.(33) coincides with the mass of the scalar field (measured in the units l_0^{-1}), the presence of these oscillations is quite natural. The domain wall with the oscillating width can be regarded as a perturbation of a proper domain wall, which by definition does not have the oscillating component.

3 Dynamics of the core in terms of local curvatures

Equation (24), which governs evolution of the core of the domain wall, has rather abstract form. In order to bring its contents to surface, we will write it in appropriately chosen local coordinates τ, σ^1, σ^2 on the world-volume Σ of the core S . We shall call them the physical coordinates.

As the τ coordinate we take the one present in formula (4), without any further specification, while the coordinates τ, σ^1, σ^2 are now defined in a vicinity of an arbitrarily chosen point $\vec{X}_0(\tau)$ of the core S in a special manner explained below. For $\vec{X} \equiv (X^1, X^2, X^3)$ in formula (4) we write

$$\begin{aligned} \vec{X}(\tau, \sigma^1, \sigma^2) &= \vec{X}_0(\tau) + \vec{e}_1(\tau)\sigma^1 + \vec{e}_2(\tau)\sigma^2 \\ &+ \frac{1}{2}\vec{e}_1(\tau) \times \vec{e}_2(\tau) \left[\frac{(\sigma^1)^2}{R_1(\tau)} + \frac{(\sigma^2)^2}{R_2(\tau)} \right] + \mathcal{O}^{(3)}(\sigma), \end{aligned} \quad (34)$$

where \times denotes the vector product, and \vec{e}_1, \vec{e}_2 are unit vectors specified below. For our purposes, the terms $\mathcal{O}^{(3)}(\sigma)$ (third order terms in σ^1, σ^2) do not have to be written explicitly. It is clear that $\sigma^1 = 0 = \sigma^2$ corresponds to the point $\vec{X}_0(\tau)$, and that the vectors $\vec{X}_{,\sigma^1}, \vec{X}_{,\sigma^2}$ tangent to S at $\vec{X}_0(\tau)$ are equal to $\vec{e}_1(\tau), \vec{e}_2(\tau)$, correspondingly. From conditions (6) we obtain that at $\vec{X}_0(\tau)$

$$n^0 = \frac{\vec{m} \cdot \dot{\vec{X}}_0}{\sqrt{1 - (\dot{\vec{X}}_0 \vec{m})^2}}, \quad \vec{n} = \frac{\vec{m}}{\sqrt{1 - (\vec{m} \dot{\vec{X}}_0)^2}}, \quad (35)$$

where

$$\vec{m} \equiv \vec{e}_1 \times \vec{e}_2 \quad (36)$$

is the unit vector normal to S at $\vec{X}_0(\tau)$ and $\dot{\vec{X}}_0 \equiv d\vec{X}_0/d\tau$. The extrinsic curvature coefficients at $\vec{X}_0(\tau)$ are equal to

$$K_{00} = \frac{\ddot{\vec{X}}_0 \vec{m}}{\sqrt{1 - (\vec{m} \dot{\vec{X}}_0)^2}}, \quad K_{0i} = K_{i0} = \frac{\dot{e}_i \vec{m}}{\sqrt{1 - (\vec{m} \dot{\vec{X}}_0)^2}},$$

$$K_{12} = 0, \quad K_{11} = \frac{1}{\sqrt{1 - (\vec{m} \dot{\vec{X}}_0)^2}} \frac{1}{R_1(\tau)}, \quad K_{22} = \frac{1}{\sqrt{1 - (\vec{m} \dot{\vec{X}}_0)^2}} \frac{1}{R_2(\tau)}, \quad (37)$$

as it follows from the definition (7). The fact that K_{12} vanishes means that the tangent vectors \vec{e}_1, \vec{e}_2 have been chosen in directions tangent to the circles of main curvatures of the surface S at the point $\vec{X}_0(\tau)$. The parametrisation (34) is the most natural local parametrisation of the core from the viewpoint of geometry of surfaces in the 3-dimensional space.

As for the choice of $\vec{X}_0(\tau)$ at different times τ , there is a large freedom due to the reparametrisation invariance of the equation (24). In physical terms, this invariance means that the core S is a "structureless" surface in the sense that translations along S are not observable. Natural choice is that $\vec{X}_0(\tau)$ is a smooth function of τ . Apart from this, $\vec{X}_0(\tau)$ for different τ 's can be chosen almost arbitrarily. Even the condition $\dot{\vec{X}}^2 < 1$ does not have to be imposed, because the core of the domain wall is merely a mathematical construct; in fact there is a numerical evidence that in a related case of

vortices the core can move with superluminal velocity, see e.g.[19]. We will use this large freedom to choose $\vec{X}_0(\tau)$ for different τ in such a manner that

$$\dot{\vec{X}}_0(\tau)\vec{e}_1(\tau) = \dot{\vec{X}}_0(\tau)\vec{e}_2(\tau) = 0, \quad (38)$$

i.e. the motion of the point $\vec{X}_0(\tau)$ is always in the direction perpendicular to S . Taking (38) into account, we can write

$$\dot{\vec{X}}_0(\tau) = v(\tau)\vec{m}. \quad (39)$$

Then, simple calculations show that Eq.(24) reduces to the following relation

$$\frac{\dot{v}(\tau)}{1 - v^2(\tau)} = \frac{1}{R_1(\tau)} + \frac{1}{R_2(\tau)}. \quad (40)$$

Thus, locally, the acceleration \dot{v} of the piece of the core is determined by the main radii of curvature, up to the Lorentz factor.

In the particular cases of a sphere ($R_1 = R_2 \equiv R$, $\dot{v} = -\dot{R}$), and of a cylinder ($R_1 = \infty$, $R_2 = R$, $\dot{v} = -\dot{R}$), the usual spherical or cylindrical coordinates, respectively, have the properties required by formula (34) and conditions (38), and (40) coincides with equations considered in [17]. In a more general case, the explicit constructing of the special coordinates (σ^1, σ^2) , such that formula (34) and condition (38) hold, might be difficult. Nevertheless, formula (40) is very helpful in qualitative analysis of motion of the core.

If we drop the gauge condition (38) while still keeping (34), then instead of formula (40) we obtain

$$\begin{aligned} \dot{v} - (\vec{m}\dot{\vec{e}}_1)(\dot{\vec{X}}_0\vec{e}_1) - (\vec{m}\dot{\vec{e}}_2)(\dot{\vec{X}}_0\vec{e}_2) = \\ [1 - (\dot{\vec{X}}_0)^2 + (\dot{\vec{X}}_0\vec{e}_2)^2]\frac{1}{R_1} + [1 - (\dot{\vec{X}}_0)^2 + (\dot{\vec{X}}_0\vec{e}_1)^2]\frac{1}{R_2}, \end{aligned} \quad (41)$$

where $v \equiv \dot{\vec{X}}_0\vec{m}$ is the perpendicular component of the velocity $\dot{\vec{X}}_0$.

Formula (40) can be reinterpreted in terms of a surface tension. To show this we pass to the rest-frame of a small piece dS of the core. (Observe that formula (40) contains the laboratory frame quantities: the time $x^0 = \tau$, the transverse velocity $v \equiv \dot{\vec{X}}_0\vec{m}$, and the radii R_1 , R_2 .) The core element

dS is parametrised by the parameters σ^1, σ^2 introduced by formula (34): $\sigma^1 = 0 = \sigma^2$ corresponds to the "center" $\vec{X}_0(\tau)$ of the element dS of the core. The radii R_1, R_2 are now the rest-frame curvature radii, denoted by R_1^{rest}, R_2^{rest} . For simplicity, we restrict our considerations to the case when the piece dS of the core neither rotates nor is deformed at the chosen instant of time. Then $\vec{e}_1 = \vec{e}_2 = 0$ and $\vec{m} = 0$. Let us assume that on a piece \vec{dl} of the boundary $\partial(dS)$ of dS acts a force $d\vec{F}$ of the magnitude ωdl ($dl \equiv |\vec{dl}|$), tangent to the core and perpendicular to \vec{dl} , directed to the outside of dS . Here ω is a constant. It is easy to see that

$$d\vec{F} = \omega \vec{N}(\sigma^1, \sigma^2) \times \vec{dl},$$

where

$$\vec{N}(\sigma^1, \sigma^2) \equiv \frac{\vec{X}_{,\sigma^1}^{rest} \times \vec{X}_{,\sigma^2}^{rest}}{|\vec{X}_{,\sigma^1}^{rest} \times \vec{X}_{,\sigma^2}^{rest}|}$$

is a unit vector perpendicular to the core at \vec{dl} . In the last formula, the vectors $\vec{X}_{,\sigma^1}^{rest}, \vec{X}_{,\sigma^2}^{rest}$ are the vectors tangent to S at the point $\vec{X}(\tau, \sigma^1, \sigma^2)$, calculated from the rest frame counterpart of formula (34), i.e.

$$\vec{X}_{,\sigma^1}^{rest} = \vec{e}_1 + \frac{\sigma^1}{R_1^{rest}} \vec{m}, \quad \vec{X}_{,\sigma^2}^{rest} = \vec{e}_2 + \frac{\sigma^2}{R_2^{rest}} \vec{m}. \quad (42)$$

Notice that $\vec{e}_i^{rest} = \vec{e}_i$ – the vectors tangent to dS at $\vec{X}_0(\tau)$ – do not change under the boost in the perpendicular direction given by \vec{m} . The same is true for the parameters σ^1, σ^2 . Next step is to calculate the integral

$$\int_{\partial(dS)} d\vec{F},$$

which gives the total force acting on dS . In the limit dS shrinking to the point $\vec{X}_0(\tau)$, the result for the integral is

$$\vec{f}_{rest} = \omega \left(\frac{1}{R_1^{rest}} + \frac{1}{R_2^{rest}} \right) |dS| \vec{m},$$

where $|dS|$ denotes the area of dS . (In fact, it is the dominating contribution only; we have neglected terms which vanish faster than $|dS|$.) In order to

find the corresponding force in the laboratory frame, we apply the boost with velocity $v \equiv |d\vec{X}_0/dt|$ in the direction \vec{m} : $\vec{f}_{lab} = (\sqrt{1-v^2})^{-1/2} \vec{f}_{rest}$, i.e.

$$\vec{f}_{lab} = \omega |dS| \frac{1}{\sqrt{1-v^2}} \left(\frac{1}{R_1^{rest}} + \frac{1}{R_2^{rest}} \right) \vec{m}.$$

The laboratory frame curvature radii are related to the rest frame ones by the formula

$$R_i = \frac{R_i^{rest}}{\sqrt{1-v^2}}, \quad i = 1, 2.$$

Notice that R_i are bigger than R_i^{rest} – this is due to the fact that in the laboratory frame the surface S is flattened in the direction of motion by Lorentz contraction. Another way to obtain this transformation law is to use the formulae (37) and the fact that K_{11}, K_{22} at $\sigma^1 = \sigma^2 = 0$ are invariant under Lorentz boosts in the \vec{m} direction. Relativistic Newton equation of motion in the laboratory frame has the following form

$$\omega |dS| \frac{d^2 \vec{X}_0}{ds^2} = \frac{\omega |dS|}{1-v^2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \vec{m}, \quad (43)$$

where s denotes the proper time. We have assumed that the rest-frame "mass" of the piece of the core comes entirely from the surface tension and is equal to $\omega |dS|$. Equation (43) reduces to relation (40): one should relate the s variable to the laboratory time τ ($ds = \sqrt{1-v^2} d\tau$), to substitute $d\vec{X}_0/ds = v \vec{m}/\sqrt{1-v^2}$, and to use the assumption $\dot{\vec{m}} = 0$. If the piece dS rotates or is subjected to a deformation, the above presented reasoning should be generalized by introducing an appropriate stress tensor for the core. We will not dwell on this.

At the end of the previous Section we have obtained the first correction to the inverse dimensionless width of the domain wall in the co-moving frame. In the absence of the oscillatory component it is equal to $N^{(1)}$ given by formula (32). The quantity $K_b^a K_a^b$ present in that formula can easily be calculated in the physical coordinates (σ^1, σ^2) defined by formula (34). We do not assume here the gauge (38). We obtain the following result: at the point $\vec{X}_0(\tau)$

$$\frac{\xi_0}{2l_0} \approx 1 + l_0^2 K_b^a K_a^b, \quad (44)$$

where

$$K_b^a K_a^b = \frac{2}{1-v^2} \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_1 R_2} \right) - \frac{2}{(1-v^2)^2} \left[\left(\dot{m}_1 + \frac{v_1}{R_1} \right)^2 + \left(\dot{m}_2 + \frac{v_2}{R_2} \right)^2 \right]. \quad (45)$$

Here we have introduced a short notation for components of the velocity and of the vector \vec{m} :

$$v_i = \vec{X}_0 \vec{e}_i, \quad \dot{m}_i = \dot{\vec{m}} \vec{e}_i = -\vec{m} \dot{\vec{e}}, \quad i = 1, 2; \quad v \equiv \vec{X}_0 \vec{m}. \quad (46)$$

When the conditions (38) are satisfied, in formula (45) one should put $v_i = 0$.

Let us present some implications of the formula (45). First, if each piece of the core is at rest at certain instant of time, then (at this moment) $v_i = 0 = \dot{m}_i$, and therefore formulae (44),(45) give

$$\frac{\xi_0}{2l_0} \approx 1 + 2l_0^2 \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_1 R_2} \right). \quad (47)$$

Thus, bending the domain wall is associated with making it thicker (if we do not excite the oscillatory component).

If a piece of the core has a nonzero velocity at certain instant of time, then the terms in the square bracket present on the r.h.s. of formula (45) may compensate the effect of non-zero curvature because of the minus sign. To check this possibility, we have considered the following class of exact solutions of Eq.(24):

$$\vec{X}(x^1, x^2, \tau) = \begin{pmatrix} x^1 \\ x^2 \\ f(l_1 x^1 + l_2 x^2 - \tau) \end{pmatrix}, \quad (48)$$

where l_1, l_2 are constants obeying the condition $l_1^2 + l_2^2 = 1$, and f can be any smooth function. This plane wave type solution gives a membrane "levitating" over the (x^1, x^2) plane on the altitude $x^3 = f(l_1 x^1 + l_2 x^2 - \tau)$, with an infinite plane wave of constant shape propagating as a whole along the membrane in the direction (l_1, l_2) with the velocity of light. The fact that such solutions exist can be deduced from a particular domain wall solution found in [20]. Straightforward computations give for the solution (48)

$$\frac{1}{R_2} = 0, \quad \frac{1}{R_1} = \frac{f''}{(1+f'^2)^{3/2}}, \quad v = -\frac{f'}{\sqrt{1+f'^2}}, \quad (49)$$

$$\dot{m}_1 = \frac{f''}{1+f'^2}, \quad \dot{m}_2 = 0, \quad v_1 = -\frac{f'^2}{\sqrt{1+f'^2}}, \quad v_2 = 0. \quad (50)$$

It follows from formula (45) that in this case $N^{(1)} = 0$. Thus, in the leading order the presence of the plane wave travelling along the domain wall does not influence the width of the domain wall.

4 The energy of the domain wall

The total energy and energy density are very important physical characteristics of a solution of field equations, and therefore we would like to calculate them for our domain wall solutions. The form of expression for the total energy depends on whether one integrates over the hypersurface of constant τ , or over the hypersurface of constant laboratory time x^0 . The construction of the conserved energy-momentum in the former case has been presented in Section V of the paper [17]. In the present paper we consider the standard energy, i.e. the one obtained by integration over the hyperplane of constant x^0 .

Because the total energy of the field is constant during the motion of the domain wall, we can calculate it at an arbitrarily chosen laboratory frame time x^0 . Of course, in order to really have the energy constant in time one should use the exact solution of the field equation (2). It will not come out exactly constant if we calculate it for an approximate, time dependent solution, in particular for the one given by formulae (18).

The energy-momentum tensor in our model has the following components

$$T^{\mu\nu} = \partial^\mu \Phi \partial^\nu \Phi + \eta^{\mu\nu} L, \quad (51)$$

with L given by formula (1). The lab-frame energy E is given by the integral

$$E = \int d^3x T^{00}. \quad (52)$$

In this formula the field Φ , which is a scalar with respect to coordinate transformations, can be regarded as a function of the $(\tau, \sigma^1, \sigma^2, \xi)$ variables. With the help of a formula for differentiation of composite functions T^{00} can

be written in the following form

$$T^{00} = [(M^{-1})^{0a}\Phi_{,a} + n^0\Phi_{,\xi}]^2 + \frac{1}{2}g_{ab}(M^{-1})^{ac}(M^{-1})^{bd}\Phi_{,c}\Phi_{,d} + \frac{1}{2}(\Phi_{,\xi})^2 + V(\Phi), \quad (53)$$

where $a = 0, 1, 2$; n^0 is the $\mu = 0$ component of the four-vector (n^μ) ; the potential

$$V(\Phi) = \frac{\lambda}{2}(\Phi^2 - \Phi_0^2)^2, \quad (54)$$

and the matrix $[M^{ab}]$ (the inverse of $[M_{ab}]$) is given by formula (16).

Also the volume element d^3x is expressed by the co-moving coordinates,

$$d^3x = \sqrt{-g} \frac{h(\tau, \sigma^1, \sigma^2, \xi)}{1 - \xi K_{0a} g^{a0}} d\xi d\sigma^1 d\sigma^2. \quad (55)$$

The integration over the ξ coordinate is effectively restricted to the interval $(\xi_0, -\xi_1)$, because in the vacuum $T^{00} = 0$. Observe that on the hyperplane of constant x^0 the τ variable in general becomes a function of ξ ; this is because

$$\tau + \xi n^0(\tau, \sigma^i) = x^0 = \text{constant}. \quad (56)$$

Thus, constant x^0 does not correspond to constant τ , except for the core where $\xi = 0$ and $x^0 = \tau$. This complicates very much calculation of the integral over ξ because in general we do not know the explicit form of the τ dependence – for this one would have to know explicit solutions of Eq.(24).

There is a particular case in which the calculation of the energy is relatively simple: when each piece of the core of the domain wall is at rest at the initial time x^0 . Mathematically, this means that the first derivative of \vec{X} with respect to x^0 vanishes. Then, one can show that $n^0 = 0$, hence $x^0 = \tau$ for all ξ .

In the following part of this Section we will use the physical coordinates defined by formula (34) and the gauge condition (38). In these coordinates $d\sigma^1 d\sigma^2 = |dS|$, the area of the infinitesimal element dS of the core. For the core at rest $v = 0$ and $\dot{\vec{e}}_i = 0$. This implies that

$$d^3\vec{x} = (1 - \frac{\xi}{R_1})(1 - \frac{\xi}{R_2})d\xi|dS|, \quad (57)$$

and

$$T^{00} = \frac{1}{2[1 + \xi(\frac{1}{R_1} + \frac{1}{R_2})]^2} (\Phi_{,\tau})^2 \quad (58)$$

$$+ \frac{1}{2(1 - \frac{\xi}{R_1})^2} (\Phi_{,\sigma^1})^2 + \frac{1}{2(1 - \frac{\xi}{R_2})^2} (\Phi_{,\sigma^2})^2 + \frac{1}{2} \Phi_{,\xi}^2 + V(\Phi).$$

Here we have also used the relation (40). Therefore, formulae (57), (58) are valid for the Nambu-Goto domain walls only.

The next step is to integrate T^{00} over ξ in the interval $(\xi_0, -\xi_1)$. The integrand is a rational function of ξ . There is no danger of a vanishing denominator, because we have assumed that the width of the domain wall ($2\xi_0$) is smaller than the curvature radii R_i , otherwise the region of validity of the co-moving coordinates would be too narrow to cover the whole width of the domain wall.

For Φ we take the approximate solution (18) with (22), (23) taken into account: for $-\xi_0 \leq \xi \leq \xi_0$

$$\Phi(\tau, \sigma^1, \sigma^2, \xi) = \frac{3}{2} \Phi_0 \left(\frac{\xi}{\xi_0} - \frac{1}{3} \frac{\xi^3}{\xi_0^3} \right). \quad (59)$$

We may write

$$\Phi_{,\tau} \equiv \frac{\Phi_0}{\xi_0} \frac{\partial \xi_0}{\partial \tau} f_0(z), \quad \Phi_{,\sigma^i} \equiv \frac{\Phi_0}{\xi_0} \frac{\partial \xi_0}{\partial \sigma^i} f_0(z), \quad (60)$$

$$\Phi_{,\xi} \equiv \frac{\Phi_0}{\xi_0} f_1(z), \quad V(\Phi) \equiv \frac{\lambda}{2} \Phi_0^4 f_2^2(z) = \frac{\Phi_0^2}{8l_0^2} f_2^2(z),$$

where $z \equiv \xi/\xi_0$ has values in the interval $[1, -1]$, and the dimensionless functions $f_0(z), f_1(z), f_2(z)$ are given by the following formulae

$$f_0 = -\frac{3}{2} z(1 - z^2), \quad f_1 = \frac{3}{2} (1 - z^2), \quad f_2 = \frac{9}{4} z^2 (1 - \frac{1}{3} z^2)^2 - 1. \quad (61)$$

The energy $E = \int d^3x T^{00}$ is given by formula

$$E = \frac{1}{2} \Phi_0^2 \int_{the \ core} |dS| \int_{-1}^1 dz \left\{ \left[\frac{(1 - z \frac{\xi_0}{R_1})(1 - z \frac{\xi_0}{R_2})}{[1 + z(\frac{\xi_0}{R_1} + \frac{\xi_0}{R_2})]^2} \left(\frac{\partial \xi_0}{\partial \tau} \right)^2 \right. \right.$$

$$\begin{aligned}
& \left[\frac{1 - z \frac{\xi_0}{R_2}}{1 - z \frac{\xi_0}{R_1}} \left(\frac{\partial \xi_0}{\partial \sigma^1} \right)^2 + \frac{1 - z \frac{\xi_0}{R_1}}{1 - z \frac{\xi_0}{R_2}} \left(\frac{\partial \xi_0}{\partial \sigma^2} \right)^2 \right] \frac{1}{\xi_0} f_0^2(z) \\
& + \frac{1}{2l_0} \left(1 - z \frac{\xi_0}{R_1} \right) \left(1 - z \frac{\xi_0}{R_2} \right) \left[\frac{2l_0}{\xi_0} f_1^2(z) + \frac{\xi_0}{2l_0} f_2^2(z) \right] \Bigg\}, \quad (62)
\end{aligned}$$

where, by the assumption, $\xi_0/R_i \ll 1$. Let us recall that this formula is valid for domain walls with the core at the instant rest and obeying the Nambu-Goto equation (24). Up to this point we have not used the approximate solutions for the half-width ξ_0 following from the Ansatz (29).

Let us point out two particular consequences of formula (62). First, we see that all nonuniformities of the width of the domain wall increase the energy density because nonvanishing derivatives $\partial \xi_0 / \partial \sigma^i$ always give positive contribution to the integrand. Also τ -dependence of ξ_0 increases the energy.

Second, there is a curvature-dependent energy associated with the width of the domain wall which is present even in the case of constant width. It is given by the second term on the r.h.s. of formula (62) (the one with f_1, f_2).

For the solution (59) the dependence on z is explicit, and it is not difficult to perform the integration over z . Because the resulting formula is lengthy we will not present it here in the general case. Let us calculate the energy in the particular case, when the oscillatory component is absent, the half-width ξ_0 is constant, and each piece of the core is at instant rest. Then, ξ_0 is related to the curvatures by formulae (44),(45), with $v = v_i = \dot{m}_i = 0$. Because the core is at the instant rest, the curvature radii have vanishing derivatives with respect to τ , and therefore $\partial \xi_0 / \partial \tau = 0$. Formula (44) for $\xi_0/2l_0$ is approximate, so we shall calculate E with the same accuracy, i.e. to the second order in l_0/R_i . We obtain the following result

$$\begin{aligned}
E = \frac{\Phi_0^2}{2l_0} \int_{the \ core} |dS| & \left[\frac{1}{2} (c_1 + c_2) - l_0^2 (c_1 - c_2) \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_1 R_2} \right) \right. \\
& \left. + 2l_0^2 (d_1 + d_2) \frac{1}{R_1 R_2} \right], \quad (63)
\end{aligned}$$

where the constants c_i, d_i are defined as follows

$$\begin{aligned}
c_1 & \equiv \int_{-1}^1 dz f_1^2(z) = \frac{12}{5}, \quad c_2 \equiv \int_{-1}^1 dz f_2^2(z) \approx 0.777, \\
d_1 & \equiv \int_{-1}^1 dz z^2 f_1^2(z) = \frac{12}{35}, \quad d_2 \equiv \int_{-1}^1 dz z^2 f_2^2(z) \approx 0.066. \quad (64)
\end{aligned}$$

Immediate consequence of formulae (63),(64) is that domain walls with nonzero extrinsic curvature can have smaller energy. This is because $c_1 - c_2$ has come out positive. For instance, a straight infinite cylindrical domain wall with the core radius R (in this case $1/R_1 = 0, 1/R_2 = 1/R$) has smaller energy per unit area than the planar domain wall (for which $1/R_1 = 0, 1/R_2 = 0$). Thus, the domain wall prefers to have wrinkles. Let us recall that this has been found with the help of the approximate solution. It remains to be checked whether $c_1 - c_2$ is positive for the corresponding exact solution of Eq.(2). For this reason we do not claim that this is a proven result – it is just an indication, to be checked in another investigation.

Another interesting problem, namely finding local minima of the energy E , e.g. for fixed area $|S|$ of a compact core with a given genus, we also leave for a future investigation.

5 The axially symmetrical toroidal domain wall

In this Section we shall apply the presented formalism to a toroidal domain wall. Our motivation for doing this is that such domain walls are next to planar, cylindrical or spherical ones with respect to complexity of their geometry, and to our best knowledge they have not been investigated as yet. Cylindrical or spherical domain walls have been considered in, e.g., [21],[9],[13],[17]. We shall consider the simplest case of a toroidal domain wall, characterised by the axial symmetry with respect to rotations around the x^3 -axis.

We shall parametrise the toroidal core of our domain wall (at each instant of time) by two angles, $\phi \in [0, 2\pi]$ and $\theta \in [0, 2\pi]$:

$$\vec{X}(\tau, \theta, \phi) = \begin{pmatrix} (R(\tau) + r(\tau, \theta)\cos\theta) \cos\phi \\ (R(\tau) + r(\tau, \theta)\cos\theta) \sin\phi \\ r(\tau, \theta) \sin\theta \end{pmatrix}, \quad (65)$$

where τ is the laboratory frame time as introduced by formula (4), ϕ is the azimuthal angle in the (x^1, x^2) plane, and θ is the angle parametrising cross sections C of the torus with the half-planes of constant ϕ . C does not have to be a circle. Because of the axial symmetry, the cross sections C are identical for all angles ϕ . The radius $R(\tau)$ gives the distance from the x^3 -axis to a

"central" circle of the torus. Only if the cross sections C are circular there is a natural choice for $R(\tau)$: the distance from the x^3 -axis to the centers of the circles C . In general, there is a freedom in the choice of the "central" circle; this circle is merely a mathematical construct – the physical object is the domain wall. The choice of $R(\tau)$ has influence on the form of the $r(\tau, \theta)$ function. In the following we choose

$$R(\tau) = \text{constant} \equiv R_0. \quad (66)$$

Thus, we have to calculate only one function $r(\tau, \theta)$. For correctness of the parametrisation (65) it is required that $r(\tau, \theta) > 0$ and $R_0 > r(\tau, \theta)$. It might happen that in order to follow evolution of the torus for a prolonged interval of time it is necessary to introduce several parametrisation patches given by (65) with different R_0 's. This is the case when after some time the "central" circle with the fixed radius R_0 turns out to lie outside of the displaced torus.

Inserting the Ansatz (65) into Eq.(24) we obtain after straightforward computations the following nonlinear equation for $r(\tau, \theta)$:

$$\begin{aligned} r(r^2 + r'^2)\ddot{r} - 2\dot{r}r'(\dot{r}'r - \dot{r}r') + (1 - \dot{r}^2)(r^2 + 2r'^2 - rr'') \\ + (r^2 + r'^2 - \dot{r}^2r^2)\left(1 - \frac{R_0 - r'\sin\theta}{R_0 + r\cos\theta}\right) = 0, \end{aligned} \quad (67)$$

where $\dot{r} \equiv \partial r(\tau, \theta)/\partial \tau$, $r' \equiv \partial r(\tau, \theta)/\partial \theta$.

In general, equation (67) is not identical with formula (40) – they are written in different coordinates. The point is that \vec{X} given by formula (65), when expanded in $\theta - \theta_0, \phi - \phi_0$ in a vicinity of a point (θ_0, ϕ_0) , does not have the form (34), and also conditions (38) are not satisfied, in general. In other words, reparametrisation gauge fixing implied by the formulae (65), (66) is different from the gauge fixing implied by the formula (34) and condition (38). Equation (67) can of course be transformed to the form (40), but this transformation involves nontrivial changes of coordinates.

For simplicity, in the following part of this Section we shall consider only tori which at certain initial instant τ_0 have circular cross sections. Such tori, in addition to the axial symmetry have also a reflection symmetry $x^3 \rightarrow -x^3$. We also assume that for $\tau = \tau_0$ each point of the torus has zero velocity. This case is sufficient in order to get a feeling about evolution of the toroidal domain walls.

One may ask whether the initially circular cross section will stay circular at later times. To investigate this, we recalculate the basic equation (24) for the Ansatz (65) without the assumption (66) that R is constant in time. If the cross section C stays circular, there should exist such a choice of the function $R(\tau)$ that the solution r is constant in θ . We have found that such a choice is not possible. Therefore, the initially circular cross section will always be deformed during evolution of the torus.

Qualitative picture of the motion governed by Eq.(67) can be obtained by considering motion of the outermost and innermost circles on the core, $\theta = 0$ and $\theta = \pi$, respectively. Because of the symmetries of the torus, $r' = 0$ for $\theta = 0, \pi$, for all times. Therefore, for $\theta = 0$ Eq.(67) reduces to

$$\frac{\ddot{r}}{1 - \dot{r}^2} = -\left(\frac{1}{r} - \frac{r''}{r^2}\right) - \frac{1}{R_0 + r}. \quad (68)$$

Similarly, for $\theta = \pi$ we have

$$\frac{\ddot{r}}{1 - \dot{r}^2} = -\left(\frac{1}{r} - \frac{r''}{r^2}\right) + \frac{1}{R_0 - r}. \quad (69)$$

One can check that equations (68), (69) coincide with formula (40). The main radii of curvature of the torus at the point (θ, ϕ) are given by the formulae

$$\frac{1}{R_1} = \frac{1}{\sqrt{r^2 + r'^2}} \frac{r' \sin \theta + r \cos \theta}{R_0 + r \cos \theta}, \quad (70)$$

$$\frac{1}{R_2} = -\frac{rr'' - 2r'^2 - r^2}{(r^2 + r'^2)^{3/2}}. \quad (71)$$

Because $r' = 0$ for $\theta = 0, \pi$, the curvature radii are, respectively, $R_1 = \pm(R_0 \pm r)$, $R_2^{-1} = 1/r - r''/r^2$ (we choose the convention that the normal \vec{m} to the torus is directed inwards). Also $\dot{\vec{X}}$ does not have the tangent components for $\theta = 0, \pi$, and $v = -\dot{r}$ in the both cases. Now it is easy to check that Eqs.(68), (69) follow directly from formula (40).

Equations (68), (69) imply that there are two classes of tori, differing by their motion. To see this, let us consider Eqs.(68),(69) at the initial time τ_0 . Then C is a circle. Let us denote its radius by r_0 . It is clear that $r'_0 = r''_0 = 0$. Also, by the assumption, $\dot{r} = 0$ at the initial time. Therefore, on the r.h.s. of Eq.(68) we have the force

$$-\frac{1}{r_0} - \frac{1}{R_0 + r_0},$$

which is always negative. It follows that the outermost circle " $\theta = 0, \phi - \text{variable}$ " will always move towards the x^3 -axis. As for Eq.(69), the force is equal to

$$-\frac{1}{r_0} + \frac{1}{R_0 - r_0},$$

and it is negative only for $r_0 < R_0/2$. For $r_0 = R_0$ it vanishes, while for $r_0 > R_0/2$ it is positive. Therefore, in this last case the circle $\theta = \pi$ will start to move away from the x^3 -axis. To summarize, when the initial circular cross section C of the torus is small enough, the torus will shrink towards a circle in the (x^1, x^2) -plane with the center on the x^3 -axis. On the other hand, if the initial circular cross section is sufficiently large, we expect that the torus will start to shrink towards the x^3 -axis. As for the intermediate case of $r_0 = R_0/2$, initially the acceleration of the innermost circle " $\theta = \pi, \phi - \text{variable}$ " vanishes. At later times, as the cross section C of the torus shrinks r'' becomes negative (because r has now a local maximum at $\theta = \pi$), and on the r.h.s. of Eq.(69) the negative term r''/r_0^2 will appear. Therefore, we expect that in the case of initial data such that $r_0 = R_0/2$, $v_0 = 0$, the circle $\theta = \pi$ will start to move away from the x^3 axis. Numerical solutions of equation (67) confirm this expectations, see Figs. 1a÷1c.

Till now we have discussed evolution of the core of the toroidal domain wall. From formulae (44), (45) we obtain approximate value of the width of the toroidal domain wall in the absence of the oscillatory component. This is especially simple for $\theta = 0, \pi$, because there $\vec{m} = 0$ and $v_1 = v_2 = 0$. We find that

$$\frac{\xi_0}{2l_0} \approx 1 + \frac{2l_0^2}{1 - v^2} \left[\frac{1}{r^2} \left(1 - \frac{r''}{r} \right)^2 + \frac{1}{(R_0 \pm r)^2} \pm \frac{1}{r(R_0 \pm r)} \right], \quad (72)$$

where the \pm signs are for $\theta = 0, \pi$, correspondingly. From these formulae one can see that for r/R_0 small enough (a slim ring far away from the x^3 -axis) the outer side ($\theta = 0$) of the toroidal domain wall is thicker than the inner side ($\theta = \pi$). On the other hand, if r/R_0 becomes closer to 1 (a fat ring close to the x^3 -axis) then, vice versa, the inner side is thicker than the outer one (when the oscillatory component Ω is absent).

As for evolution of the symmetric toroidal domain wall at later times, when the rest-frame curvature radii become too small for the Ansatz (18) to be a reasonable approximation, one can make some guesses based on

results of numerical computations, [13]. We expect that the slim torus will completely disperse into radiation after the first collapse to a circle in the (x^1, x^2) -plane. The fat torus will first change its topology from toroidal to spherical one – this will happen when the innermost circle of the torus is shrunk to the origin $x^1 = x^2 = x^3 = 0$. In the process some energy will be lost into radiation. Next, the domain wall will evolve as a deformed sphere. Eventually it will collapse to the origin, and it will disperse into radiation. This scenario might be different if we pass to another model. As pointed out in [13], sine-Gordon domain walls can reemerge after the collapse. In this case the domain walls can pass through each other with only a partial loss of energy into the radiation, and therefore one can expect that the torus collapses and reemerges several times before it follows the previous scenario in a final collapse. It has also been noticed in the paper [13] that cylindrical sine-Gordon domain walls do not bounce in contradistinction to the spherical ones. This suggests that whether the sine-Gordon torus bounces or disappears after the first collapse might also depend on the big radius of the torus. At the initial time it is equal to R_0 , and the curvature $1/R_1$ depends on it, see formula (70). For large R_0 the torus locally is like a straight cylinder, while for small R_0 curvature is more pronounced and such torus might behave more like a spherical domain wall. Probably the only way to verify these scenarios is to perform numerical calculations.

6 Remarks

(a) Calculations presented in Section 2 of this paper show that the method proposed in [17] for cylindrical and spherical domain walls can be applied also in the more general case. The resulting equation (24) for the core of the domain wall is equivalent to the Nambu-Goto equation (25). Our method seems to be quite universal. Change in the basic equation (2) would result only in different form of the recurrence relations (19),(20). For example, if we change the quartic potential V (formula (54)) to the sine-Gordon type then only numerical coefficients in (19),(20) are different. The polynomial approximation can also be applied to non-relativistic domain walls in condensed matter physics.

We have not obtained any terms with higher derivatives in the equation for the core. Such terms are usually related to so called extrinsic curvature

corrections in the effective action for the core, [9]. Nevertheless, we would not conclude that the pure Nambu-Goto equation for the core is the exact, final answer for all domain walls with small curvature, for two reasons. First, our solution is approximate one, and a better approximation could reveal corrections to the Nambu-Goto equation. Second, the basic field equation (2) possesses infinitely many solutions in the topological class of the single domain wall. Our cubic polynomial Ansatz probably picks only a subset of these solutions. There might be other solutions for which the core would not obey the pure Nambu-Goto equation (25).

Actually, it is easy to point out possible ways to improve our approximation and to find more general domain wall solutions. Natural step to find a wider class of solutions within the cubic polynomial approximation is to divide the interval $(\xi_0, -\xi_1)$ (see formula (18)) into two subintervals, and to use two independent cubic polynomials in each of them. Next, one should match smoothly the two polynomials with each other and with the vacuum fields to obtain continuous Φ and $\partial_\xi \Phi$. In order to improve our approximation one could use polynomials of higher order, as suggested by considerations presented in the Appendix. Work along these lines is in progress.

(b) Formula (40) from Section 3, giving the transverse acceleration in terms of the local curvature radii, is very useful in providing qualitative understanding of time evolution of the core. For example, one could foresee from it that a bump on otherwise almost flat core will first split into two bumps travelling in the opposite directions, next more smaller bumps will appear and finally the bump will disperse into small ripples. Numerical computations we have carried out in the case of an axially symmetric bump on the toroidal core described by equation (67) confirm this expectation.

(c) Let us end this Section by mentioning several possibilities to extend our work. The first one is to improve our description of dynamics of the domain walls by including possible radiation, creation of pairs "domain wall + anti-domain wall", and interactions between close parts of the domain wall.

Second, one would like to have an analytical (approximate) description of domain walls also in the case when the curvature radii of the core are comparable with the width of the domain wall. Our results are for the case when the radii are much larger than the width.

Third, one could try to apply the polynomial approximation to curved, non-static vortices in relativistic field theories as well as in condensed matter

physics. We have seen that this approximation is capable of yielding rather detailed information about dynamics of the width of the domain wall. It would be very interesting to have such an information also for vortices.

THE APPENDIX. The accuracy of the polynomial approximation

Here we would like to comment on accuracy of the approximation used in this paper. It is instructive to apply the above presented scheme to the planar domain wall given by the exact solution (3). Then,

$$K_{ab} = 0, \quad (g_{ab}) = \text{diag}(-1, 1, 1), \quad \xi = x^3,$$

and in the absence of the oscillating component $\tilde{A} = 1$, i.e. $\xi_0 = 2l_0$. Then

$$\Phi(x^3) = \Phi^{(1)}(x^3) \equiv \frac{3}{4}\Phi_0 \frac{x^3}{l_0} \left(1 - \frac{1}{12}\left(\frac{x^3}{l_0}\right)^2\right). \quad (73)$$

The boundaries of this domain wall are at $x^3 = \pm 2l_0$. Comparing $\Phi^{(1)}$ with the solution (3), we see that it is not equal to the first two terms of Taylor expansion of the solution (3). Thus, we do not recover the exact solution (3) term by term in the Taylor expansion. On the other hand, the approximate solution $\Phi^{(1)}$ has the right global characteristics of the domain wall like energy per unit area or the boundary conditions. The energy per unit area in the case of solution (3) is

$$E_0 = \frac{2}{3} \frac{1}{l_0} \Phi_0^2.$$

It is minimal in the topological class of single domain wall, [22]. The energy of the solution $\Phi^{(1)}$ is a little bit higher

$$E^{(1)} \approx 0.79 \frac{1}{l_0} \Phi_0^2.$$

For a comparison of energy densities, see Figs.2a,b.

Improving the approximation (73) consists of including terms of higher order in x^3 . We know that the exact solution is an odd function of x^3 , so the first term to be included is proportional to the fifth power of x^3 . However, it turns out that a fifth order polynomial in x^3 can not simultaneously obey

recurrence relations obtained from Eq.(2) and the continuity conditions for Φ and $\partial_{x^3}\Phi$. The reason is that the recurrence relations imply that the fifth power of x^3 comes with a positive coefficient; then it is easy to see that there can be a problem with smooth matching of the polynomial with the constant vacuum solutions.

Going to the seventh order polynomial does yield a solution. Denoting $z \equiv x^3/l_0$, the result we have obtained can be written in the form

$$\Phi^{(2)} \approx \Phi_0 (0.5750z - 0.0479z^3 + 0.0059z^5 - 0.0004z^7).$$

$\Phi^{(2)}$ smoothly matches the vacuum solutions for $x^3 \approx \pm 2.593l_0$, while for the solution $\Phi^{(1)}$ the matching takes place at $x^3 = \pm 2l_0$. Energy per unit area for the solution $\Phi^{(2)}$ is equal to

$$E^{(2)} \approx 0.706 \frac{1}{l_0} \Phi_0^2,$$

which is higher than E_0 by only 6%. Also the energy densities do not differ much from their exact values, see Figs.3a,b.

To summarize, the direct comparison presented above in the case of the planar domain wall shows that the polynomial approximation works rather well. Therefore, we expect that also for slightly curved domain walls, i.e. such that their rest-frame curvature radii are large in comparison with their width, the cubic polynomial approximation is a reasonable one, and that in order to improve it one could just use a higher order polynomial. Calculations with seventh order polynomials do not have to be cumbersome, because Taylor expansions and other necessary operations can be carried out by a computer.

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Figure captions

Fig.1a. Numerical solution of equation (67). The initial data are $r(0, \theta) = 0.4$, $\dot{r}(0, \theta) = 0$. The contours are the cross sections C of the torus with, e.g., (x^1, x^3) plane. They are determined by the function $r(t, \theta)$ (with θ changing from 0 to 2π). The cross sections are shown for the following values of $t \equiv \tau/R_0$: 0.00, 0.15, 0.25, 0.35, 0.45, 0.55. The horizontal axis shows the distance from the x^3 axis, measured in the unit R_0 .

Fig.1b. The same as in Fig.1a, but with the initial data $r(0, \theta) = 0.5$, $\dot{r}(0, \theta) = 0$. The values of t are: 0.00, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65.

Fig.1c. The same as in Fig.1a, but with the initial data $r(0, \theta) = 0.65$, $\dot{r}(0, \theta) = 0$. The values of t are: 0.00, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65.

Fig.2a. The comparison of the gradient contributions $1/2(\partial_{x^3}\Phi)^2$ to the energy density of the planar domain wall. The solid line corresponds to the approximate solution $\Phi^{(1)}$, the dashed line corresponds to the exact solution (3).

Fig.2b. The comparison of the potential energies $V(\Phi)$ (formula (54)) for the planar domain wall. The solid line corresponds to the approximate solution $\Phi^{(1)}$, the dashed line corresponds to the exact solution (3).

Fig.3a. The same as in Fig.2a, but for the approximate solution $\Phi^{(2)}$.

Fig.3b. The same as in Fig.2b, but for the approximate solution $\Phi^{(2)}$.

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